March 10, 1870.

WARREN DE LA RUE, Esq., Vice-President, in the Chair.

The following communications were read:—

I. "On some Elementary Principles in Animal Mechanics.—No. III. On the Muscular Forces employed in Parturition." By the Rev. SAMUEL HAUGHTON, M.D., Fellow of Trinity College, Dublin. Received January 31, 1870.

In the first stage of natural labour, the involuntary muscles of the uterus contract upon the fluid contents of this organ, and possess sufficient force to dilate the mouth of the womb, and generally to rupture the membranes. I shall endeavour to show, from the principles of muscular action already laid down, that the uterine muscles are sufficient, and not much more than sufficient, to complete the first stage of labour, and that they do not possess an amount of force adequate to rupture, in any case, the uterine wall itself.

In the second stage of labour, the irritation of the fœtal head upon the wall of the vagina provokes the reflex action of the voluntary abdominal muscles, which aid powerfully the uterine muscles to complete the second stage by expelling the fœtus. The amount of available additional force given out by the abdominal muscles admits of calculation, and will be found much greater than the force produced by the involuntary contractions of the womb itself.

The mechanical problem to be solved for both cases is one of much interest, as it is the celebrated problem of the equilibrium of a flexible membrane subjected to the action of given forces. It has been solved by Lagrange (Mécanique Analytique, p. 147) in all its generality. In the most general case of the problem, the following beautiful thereom can be demonstrated:—Let T denote the tensile strain acting in the tangential plane of the membrane, applied to rupture a band of the membrane 1 inch broad; let P denote the pressure resulting from all the forces in action, perpendicular to the surface of the membrane, and acting on a surface of one square inch; and let ρ_1 and ρ_2 denote the two radii of principal curvature of the membrane at the point considered. Then we have the following equation:—

$$P = T \times \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right).$$

If the surface, or a portion of it, become spherical, the two principal curvatures become equal, and the equation becomes

$$P = \frac{2T}{\rho}.$$

In the case of the uterus and its membranes, the force P arises from

hydrostatical pressure only, and is therefore easily measured, and the supposition of spherical curvature is approximately admissible.

It is evident from the form of the gravid uterus, that its curvature is greatest near its mouth; and the equation shows that for a given hydrostatical pressure the tensile strain is proportional to the radius of curvature; hence this strain will be greatest at the fundus of the uterus, in which part, accordingly, we find the muscular coat thicker than elsewhere. If we assume the shape of the uterus to be that of a prolate ellipsoid, whose longer diameter is 12 inches, and shorter diameter 8 inches, its mean curvature will be that of a sphere whose diameter is 9·158 inches.

The volume of the gravid uterus is found from the expression

$$\text{Volume} = \frac{4}{3}\pi a b^2 ;$$

in which a and b are the semiaxes, and π is the ratio of the circumference of a circle to its diameter; substituting for a and b their numerical values, we find the contents of the uterus to be 402·13 cubic inches.

The surface of the gravid uterus may be found from the equation

$$\text{Surface} = \frac{2\pi ab}{e}\left(\sin^{-1} e + e\sqrt{1-e^2}\right) ;$$

in which e is the excentricity of the generating ellipse. If the numerical values be substituted in this expression, it will be found that the surface of the uterus is 270·66 square inches *.

Some highly interesting conclusions may be drawn from the preceding calculations, combined with the weight of the total muscular tissue of the uterus. Heschl estimates the weight of the uterine muscles at from 1 lb. to 1·5 lb., Montgomery found the muscles of the gravid uterus to weigh 1·5 lb., and Levret estimates them at 51 cubic inches, which, with a specific gravity of 1·052, I find to be equivalent to 1·93 lb. Taking the mean of these estimates we have:—

Weight of Muscular Fibres of Gravid Womb.

	lbs.
Heschl	1·25
Montgomery	1·50
Levret	1·93
Mean	1·56

If we now suppose this quantity of muscle to be spread over the entire surface of the uterus, we find

$$\text{Mean thickness of muscular wall of uterus} \ldots \left\{\frac{1\cdot56\times7000\times1000}{252\cdot5\times270\cdot66\times1052}\right\} = 0\cdot1519 \text{ inch.}$$

* Levret estimates the contents of the gravid uterus at 408 cubic inches, and its surface at 339 square inches.

Poppel estimates the contents at 300 cubic inches, and the surface at 210 square inches.

If we suppose a ribbon, one inch in width, to be formed from the wall of the uterus, its thickness will be 0·1519 inch; and as each square inch of cross section of muscular fibre is capable of lifting 102·55 lbs., we find for the greatest tensile force producible by the contraction of the uterine muscles:—

$$\left.\begin{array}{r}\text{Tensile strain of uterine}\\ \text{wall per inch } \ldots\ldots\end{array}\right\} 102{\cdot}55 \times 0{\cdot}1519 = 15{\cdot}577 \text{ lbs.}$$

Substituting this value of T in the equation

$$P = \frac{2T}{\rho},$$

and for ρ its mean value 9·158 inches, we obtain the maximum hydrostatical pressure inside the gravid uterus that can be produced by the contraction of its muscular fibres:—

$$\left.\begin{array}{r}\text{Maximum hydrostatical pressure}\\ \text{produced by uterine contraction}\end{array}\right\} \frac{2 \times 15{\cdot}577}{9{\cdot}158} = 3{\cdot}402 \text{ lbs.}$$

This pressure, applied to a circular surface of $4\frac{1}{2}$ inches in diameter, is equal to 54·106 lbs. One hundred experiments were made by Duncan and Tait upon the hydrostatical pressure necessary to rupture the membranes which contain the liquor amnii, which are recorded in Dr. Duncan's book* (pp. 306–311). The greatest pressure observed was 3·10 lbs., and the least was 0·26 lb.; and I find that the mean rupturing pressure of all their experiments was 1·2048 lb.

Combining this experimental result with the calculation already given, of the amount of pressure producible by the muscular tissue of the womb, we may conclude that the uterine muscles are capable of rupturing the membranes in every case, and possess, in general, nearly three times the amount of force requisite for this purpose.

In the second stage of labour, the voluntary action of the abdominal muscles is called into play to aid the expulsive efforts of the uterine muscles. I have attempted to calculate the force available from the contraction of these muscles as follows.

The abdominal muscles are four in number, viz. *rectus abdominis, obliquus externus, obliquus internus* and *transversalis*. The last three muscles form curved sheets, acting upon the corresponding muscles of the opposite side by means of tendinous *aponeuroses* which meet in the linea alba, and form the sheath of the vertical *rectus abdominis* muscle. From the arrangement of all four, it is plain that the tensile force of muscular contraction in the curved wall of the belly, from the xiphoid cartilage to the symphysis pubis, is to be measured by the sum of the united forces of all the muscular sheets. If we knew the force of each muscle, and the principal curvatures of the belly in the middle line, we could calculate, by Lagrange's theorem, the hydrostatical pressure inside the abdominal cavity and available to expel fæces, urine, or a fœtus.

* Researches in Obstetrics. Edinburgh, 1868.

In order to ascertain the force of the muscles, I measured carefully their average thicknesses in three subjects, of whom one was a young woman who had borne children, and the others were men of ordinary size and appearance. The results obtained were the following:—

Thicknesses of Abdominal Muscles.

	No. 1. Male.	No. 2. Female.	No. 3. Male.
	in.	in.	in.
Rectus abdominis	0·275	0·29	0·34
Obliquus externus	0·200	0·25	0·19
Obliquus internus	0·235	0·17	0·24
Transversalis	0·127	0·15	0·14
Total	0·837	0·86	0·91

The average total thickness of the muscular walls is 0·869 inch, which is nearly identical with the measurement obtained from the female subject. It has been ascertained by careful observations, that we must add 50 per cent. to the weights of muscles in the dead subject in order to bring them to the living weights; this correction gives us 1·3035 inch for the mean thickness of the muscles causing tension in the central line of the belly, where the forces of all the muscles come into play together. Multiplying this thickness by 102·55 lbs., or coefficient of muscular contraction, we find

$$T = 1{\cdot}3035 \times 102{\cdot}55 = 133{\cdot}67 \text{ lbs.}$$

This is the tensile strain producible by the contraction of the abdominal muscles along the curved central line of the belly.

It remains now to ascertain the principal curvatures of the abdominal surface, and to use the equation

$$P = T\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

so as to determine P, the hydrostatical pressure per square inch inside the cavity of the belly, and available, either in whole or in part, for the expulsion of the fœtus during the second stage of labour.

In order to ascertain the curvature of the belly, I made experiments on three young men placed lying on their backs upon the floor, and made them depress and raise the abdominal wall as much as possible. The result was as follows:—Taking a straight line from the upper part of the symphysis pubis to the xiphoid cartilage as the fixed line of comparison, it was found possible to depress the navel one inch below this fixed line and to raise it two inches above it. When the belly was distended to the utmost by the action of the abdominal muscles, I measured the longitudinal and transverse curvatures by measuring the sagittas corresponding to a given length of tangent, with the following results:—

Number.	Diameter of longitudinal curvature.	Diameter of transverse curvature.
	in.	in.
J. G. H.	22·93	12·30
H. O.	22·73	12·80
S. H.	22·52	12·80
Mean	22·727	12·633

The curvature of the distended belly at the navel is found to be, from the foregoing measurements,

$$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{11\cdot3635}+\frac{1}{6\cdot3166}=\frac{1}{4\cdot0596}.$$

Multiplying this curvature into the tension of the abdominal muscles at the navel already found, viz. 133·67 lbs. per inch, we obtain, finally,

$$P=\frac{133\cdot67}{4\cdot0596}=32\cdot926 \text{ lbs. per square inch.}$$

This amount of expulsive force per square inch is available, although not usually employed, to assist the uterus in completing the second stage of labour. If we suppose it applied to the surface of a circle $4\frac{1}{2}$ inches in diameter (the usual width of the pelvic canal), we find that it is equivalent to 523·65 lbs. pressure.

Adding together the combined forces of the voluntary and involuntary muscles, we find—

Involuntary muscles	= 54·106	lbs.
Voluntary muscles	=523·65	,,
Total	577·75	,,

Thus we see that, on an emergency, somewhat more than a quarter of a ton pressure can be brought to bear upon a refractory child that refuses to come into the world in the usual manner*.

In order to determine by actual experiment the expulsive force of the abdominal muscles, I placed two men, of 48 and 21 years of age respectively, lying on a table upon their backs, and put a disk measuring 1·87 inch diameter just over the navel; weights were placed upon this disk and gradually increased until the extreme limit of weight that could be lifted with safety was reached; this limit was found to be in both cases 113 lbs. As the circle whose diameter is 1·87 inch has an area of 2·937 square

* The preceding result will no doubt remind the curious and well-informed reader of the statement made by Mr. Shandy, on the authority of Lithopædus Senonensis, 'De partu difficili,' that the force of the woman's efforts in strong labour pains is equal upon an average to the weight of 470 lbs. avoirdupois acting perpendicularly upon the vertex of the head of the child.

inches, the pressure perpendicular to the abdominal wall produced by the action of the abdominal muscles was

$$P=\frac{113}{2\cdot 937}=38\cdot 47 \text{ lbs. per square inch,}$$

a result which differs little from that already found by calculation from the actual measurements of the muscles and curvatures.

II. "Tables of the Numerical Values of the Sine-integral, Cosine-integral, and Exponential Integral." By J. W. L. GLAISHER, Trinity College, Cambridge. Communicated by Professor CAYLEY, LL.D. Received February 10, 1870.

(Abstract.)

The integrals

$$\int_0^x \frac{\sin u}{u}\,du, \quad \int_\infty^x \frac{\cos u}{u}\,du, \quad \int_{-x}^\infty \frac{e^{-u}}{u}\,du,$$

called the sine-integral, cosine-integral, and exponential integral, were used by Schlomilch to express the values of several more complicated integrals, and denoted by him thus,—Si x, Ci x, Ei x; the last function, however, is for all real values of x only another form of the logarithm-integral, the relation being

$$\text{Ei}\,x=\text{li}\,e^x.$$

These functions have since been shown to be the key to a very large class of definite integrals, and several hundreds have been evaluated in terms of them by Schlömilch, De Haan, &c., so that for some time they have been considered primary functions of the integral calculus, and forms reduced to dependence on them have been regarded as known.

Considering, therefore, the large number of integrals dependent on them for their evaluation, and their consequent importance as a means of extending the integral calculus, it seemed very desirable that they should be systematically tabulated, the only values which have previously been obtained being those of Si x, Ci x, Ei x, Ei $(-x)$ for the values $x=1, 2, \ldots 10$ calculated by Bretschneider, and printed in the third volume of Grunert's 'Archiv der Mathematik und Physik,' and a Table of the logarithm-integral published by Soldner at Munich in 1806.

The present Tables contain the values of Si x, Ci x, Ei x, Ei $(-x)$ for values of x from 0 to 1 at intervals of ·01 to nineteen places of decimals, for values of x from 1 to 5 at intervals of ·1, and from 5 to 15 at intervals of unity, to ten places, and for $x=20$ to twelve places. Also values of Si x and Ci x only for values of x from 20 to 100 at intervals of 5, to 200 at intervals of 10, to 1000 at intervals of 100, and for several higher values to seven places; besides Tables of the maxima and minima values of these functions, corresponding in the case of the sine-integral to multiples of π, and in the case of the cosine-integral to odd multiples of $\frac{\pi}{2}$, also to seven places.